

# AN EXTENSION OF HARNACK TYPE DETERMINANTAL INEQUALITY

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**ABSTRACT.** We revisit and comment on the Harnack type determinantal inequality for contractive matrices obtained by Tung in the sixtieth and give an extension of the inequality involving multiple positive semidefinite matrices.

*In memory of Marvin Marcus.*

In 1964, Tung [11] established the following Harnack type determinantal inequality.

**Theorem 1.** *Let  $Z$  be an  $n \times n$  complex matrix with singular values  $r_k$  that satisfy  $0 \leq r_k < 1$ ,  $k = 1, 2, \dots, n$  (i.e.,  $Z$  is a strict contraction). Let  $Z^*$  denote the conjugate transpose of  $Z$  and  $I$  be the  $n \times n$  identity matrix. Then for any  $n \times n$  unitary matrix  $U$*

$$(1) \quad \prod_{k=1}^n \frac{1 - r_k}{1 + r_k} \leq \frac{\det(I - Z^*Z)}{|\det(I - UZ)|^2} \leq \prod_{k=1}^n \frac{1 + r_k}{1 - r_k}.$$

Soon after the appearance of the Tung's paper, Marcus [8] gave another proof of (1) and pointed out that (1) is equivalent to

$$(2) \quad \prod_{k=1}^n (1 - r_k) \leq |\det(I - A)| \leq \prod_{k=1}^n (1 + r_k)$$

for any  $n \times n$  matrix  $A$  with the same singular values as the contractive matrix  $Z$ .

Marcus's proof of (2) makes use of majorization theory and singular value (eigenvalue) inequalities of Weyl. This approach is still very fruitful today in deriving determinantal inequalities; see, for example, [6, 7]. At about the same time of Marcus's proof, Hua [2] gave a proof of (2) using the determinantal inequality he had previously obtained in [3].

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2010 *Mathematics Subject Classification.* 15A45, 15A60.

*Key words and phrases.* determinantal inequality, Harnack inequality, positive semidefinite matrix.  
TO APPEAR IN LINEAR AND MULTILINEAR ALGEBRA.

**Remark 2.** We notice the following:

- (i). In the well-known book [9] of Marshall, Olkin and Arnold, Tung's theorem is cited as result E.3 on page 319 in which the condition that  $A$  be contractive is missing; that is to say, Theorem 1 need not be true in general if  $Z$  is not contractive. Take, for example,  $Z = 2iI$  with odd  $n$  (the matrix size) and appropriate  $U$ . We see neither the left nor the right inequality in (1) holds. However, the following inequality holds true for any  $n \times n$  matrix  $Z$  and any  $n \times n$  unitary matrix  $U$  (a fraction with zero denominator is viewed as  $\infty$ )

$$\prod_{k=1}^n \frac{|1 - r_k|}{1 + r_k} \leq \frac{|\det(I - Z^*Z)|}{|\det(I - UZ)|^2}.$$

- (ii). Inequalities (1) and (2) are not equivalent for general matrices. The right-hand side inequality in (2) is true for all  $n \times n$  matrices  $A$ ; that is,

$$|\det(I - A)| \leq \prod_{k=1}^n (1 + r_k).$$

Using the polar decomposition (see, e.g., [13, p. 83]), we restate and slightly generalize Theorem 1 as follows with discussions on the equality cases.

**Theorem 3.** *Let  $Z$  be an  $n \times n$  positive semidefinite matrix with eigenvalues  $r_1, r_2, \dots, r_n$ . Let  $U$  be an  $n \times n$  unitary matrix such that  $I - UZ$  is nonsingular. Then*

$$(3) \quad \prod_{k=1}^n \frac{|1 - r_k|}{1 + r_k} \leq \frac{|\det(I - Z^2)|}{|\det(I - UZ)|^2}$$

*with equality if and only if  $Z$  has an eigenvalue 1 or  $UZ$  has eigenvalues  $-r_1, -r_2, \dots, -r_n$ . If both  $Z$  and  $I - Z$  are nonsingular, the strict inequality holds for  $U \neq -I$ .*

*Moreover, if  $0 \leq r_k < 1$ ,  $k = 1, 2, \dots, n$ , then*

$$(4) \quad \frac{\det(I - Z^2)}{|\det(I - UZ)|^2} \leq \prod_{k=1}^n \frac{1 + r_k}{1 - r_k}$$

*with equality if and only if  $UZ$  has eigenvalues  $r_1, r_2, \dots, r_n$ . If  $Z$  is nonsingular, then the strict inequality in (4) holds if  $U \neq I$ .*

If we use  $\text{Spec}(X)$  to denote the spectrum of the matrix  $X$ , then equality holds in (3) if and only if  $1 \in \text{Spec}(Z)$  or  $\text{Spec}(UZ) = \text{Spec}(-Z)$ ; and equality holds in (4) if and only if  $\text{Spec}(UZ) = \text{Spec}(Z)$ .

To prove this theorem, we need a lemma which is of interest in its own right. We proceed with adoption of standard notation in majorization theory (see, e.g., [13]).

**Lemma 4.** *Let  $x = (x_1, x_2, \dots, x_n)$  and  $y = (y_1, y_2, \dots, y_n)$  be nonnegative vectors and assume that  $y$  is not a permutation of  $x$  (i.e., the multisets  $\{x_1, x_2, \dots, x_n\}$  and*

$\{y_1, y_2, \dots, y_n\}$  are not equal). Denote  $\tilde{z} = (1 + z_1, 1 + z_2, \dots, 1 + z_n)$ . We have:

$$\text{If } x \prec_{\log} y, \quad \text{then} \quad \tilde{x} \prec_{\text{wlog}} \tilde{y} \quad \text{but} \quad \tilde{x} \not\prec_{\log} \tilde{y}.$$

Consequently,

$$(5) \quad \prod_{k=1}^n (1 + x_k) < \prod_{k=1}^n (1 + y_k).$$

*Proof.* Without loss of generality, we assume that  $x$  and  $y$  are positive vectors. (Otherwise, replace the 0's in  $x$  and  $y$  by sufficiently small positive numbers and use continuity argument.) Since  $x \prec_{\log} y$ , we have  $\ln x \prec \ln y$  (see, e.g., [13, p. 344]). Let  $f(t) = \ln(1 + e^t)$ . Then  $f$  is strictly increasing and convex. By [13, Theorem 10.12, p. 342], we have  $f(\ln x) \prec_w f(\ln y)$ ; that is,  $\ln \tilde{x} \prec_w \ln \tilde{y}$ , i.e.,  $\tilde{x} \prec_{\text{wlog}} \tilde{y}$ . Since  $x$  is not a permutation of  $y$ ,  $\ln x$  is not a permutation of  $\ln y$ . Applying [13, Theorem 10.14, p. 343], we obtain  $\sum_{k=1}^n f(\ln x_k) < \sum_{k=1}^n f(\ln y_k)$ , namely,  $\sum_{k=1}^n \ln(1 + x_k) < \sum_{k=1}^n \ln(1 + y_k)$  which yields  $\prod_{k=1}^n (1 + x_k) < \prod_{k=1}^n (1 + y_k)$ .  $\square$

Under the lemma's condition, if all  $x_i$  and  $y_i$  are further in  $[0, 1)$ , a similar but reversal log-majorization inequality can be derived for  $(1 - x_k)$  and  $(1 - y_k)$ . In particular,

$$(6) \quad \prod_{k=1}^n (1 - x_k) > \prod_{k=1}^n (1 - y_k).$$

In fact, this can be proved by applying [13, Theorem 10.14, p. 343] to  $\ln x \prec \ln y$  with  $g(t) = -\ln(1 - e^t)$ , which is strictly increasing and convex when  $t \in (-\infty, 0)$ .

**Proof of Theorem 3.** We only need to show the equality cases. For (3), if  $Z$  has a singular (eigen-) value 1, then both sides vanish. If  $UZ$  has eigenvalues  $-r_1, -r_2, \dots, -r_n$ , then  $\det(I - UZ) = \prod_{k=1}^n (1 + r_k)$ . Equality is readily seen. Conversely, suppose equality occurs in (3). We further assume that no  $r_k$  ( $k = 1, 2, \dots, n$ ) equals 1. Since  $|\det(I - Z^2)| = \prod_{k=1}^n |1 - r_k|(1 + r_k)$ , we have

$$(7) \quad |\det(I - UZ)| = \prod_{k=1}^n (1 + r_k).$$

Moreover, by Weyl majorization inequality (see, e.g., [13, Corollary 10.2, p. 353]),

$$|\lambda(UZ)| \prec_{\log} \sigma(UZ) = \sigma(Z) = \lambda(Z),$$

where  $\lambda(X)$  and  $\sigma(X)$  denote the vectors of the eigenvalues and singular values of matrix  $X$ , respectively. With  $\lambda_k(X)$  denoting the eigenvalues of the  $n \times n$  matrix  $X$ ,  $k = 1, 2, \dots, n$ , by the lemma, we have

$$0 < |\det(I - UZ)| = \prod_{k=1}^n |1 - \lambda_k(UZ)| \leq \prod_{k=1}^n (1 + |\lambda_k(UZ)|) \leq \prod_{k=1}^n (1 + r_k).$$

Thus, (7) yields  $|1 - \lambda_k(UZ)| = 1 + |\lambda_k(UZ)|$  for all  $k$ , which implies  $\lambda_k(UZ) \leq 0$  for  $k = 1, 2, \dots, n$ , i.e., all eigenvalues of  $-UZ$  are nonnegative. If  $|\lambda(UZ)| = \lambda(-UZ)$  is

not a permutation of  $\lambda(Z)$ , then, by strict inequality (5), we have  $\prod_{k=1}^n (1 + |\lambda_k(UZ)|) < \prod_{k=1}^n (1 + \lambda_k(Z)) = \prod_{k=1}^n (1 + r_k)$ , a contradiction to (7). It follows that  $UZ$  has the eigenvalues  $-r_1, -r_2, \dots, -r_n$ .

For the equality in (4), it occurs if and only if  $\prod_{k=1}^n (1 - r_k) = |\det(I - UZ)|$ . Note that  $|\lambda(UZ)| \prec_{\log} \sigma(Z)$  and

$$(8) \quad \prod_{k=1}^n |1 - \lambda_k(UZ)| \geq \prod_{k=1}^n (1 - |\lambda_k(UZ)|) \geq \prod_{k=1}^n (1 - \sigma_k(Z)) = \prod_{k=1}^n (1 - r_k).$$

The first equality in (8) occurs if and only if all  $\lambda_k(UZ)$  are in  $[0, 1]$ ; the second equality occurs if and only if  $\lambda(UZ)$  is a permutation of  $\sigma(Z)$ , i.e.,  $\text{Spec}(UZ) = \text{Spec}(Z)$ .

Now assume that  $Z$  is nonsingular and suppose that equality holds in (4). Then  $UZ$  has eigenvalues  $r_1, r_2, \dots, r_n$ . Moreover, the singular values of  $UZ$  are  $r_1, r_2, \dots, r_n$ . Let  $P = UZ$ . Then the eigenvalues of  $P$  are just the singular values of  $P$ . So  $P$  is positive definite. It follows that  $U = PZ^{-1}$  has only positive eigenvalues. Since  $U$  is unitary,  $U$  has to be the identity matrix. The case for (3) is similar.  $\square$

In what follows, we prove an extension of Theorem 1 that involves multiple matrices.

**Theorem 5.** *Let  $Z_i, i = 1, 2, \dots, m$ , be  $n \times n$  positive semidefinite matrices. Suppose that the eigenvalues of  $Z_i$  are  $r_{ik}$  satisfying  $0 \leq r_{ik} < 1, k = 1, 2, \dots, n$ . Then for any  $n \times n$  unitary matrix  $U$  and positive scalars  $w_i, i = 1, 2, \dots, m, \sum_{i=1}^m w_i = 1$ , we have*

$$(9) \quad \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i}.$$

*Equality on the left-hand side occurs if and only if all  $Z_i$  are equal to  $Z$ , say, and  $Z$  has an eigenvalue 1 or  $\text{Spec}(UZ) = \text{Spec}(-Z)$  (in which  $U = -I$  if  $Z$  is nonsingular); Equality on the right-hand side occurs if and only if all  $Z_i$  are equal to  $Z$ , say, and  $\text{Spec}(UZ) = \text{Spec}(Z)$  (in which  $U = I$  if  $Z$  is nonsingular).*

Clearly, if  $m = 1$ , then (9) reduces to (1).

*Proof.* We begin by noticing the fact that for  $n \times n$  Hermitian matrices  $A$  and  $B$ , if  $\lambda_k(A + B) = \lambda_k(A) + \lambda_k(B)$  for all  $k = 1, 2, \dots, n$ , where  $\lambda_k(X)$  denotes the  $k$ th largest eigenvalue of  $X$ , then  $A$  and  $B$  are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonal in the same order. (The converse is also true.) This is easily seen from [10, Theorem 2.4]. The result holds for multiple matrices.

Our additional ingredients in proving (9) are Fan's majorization relation that  $\lambda(H + S) \prec \lambda(H) + \lambda(S)$  for  $n \times n$  Hermitian matrices  $H$  and  $S$  (see, e.g., [13, p. 356]) and Lewent's inequality [4, 5] which asserts, for  $x_i \in [0, 1], i = 1, 2, \dots, n$ ,

$$\frac{1 + \sum_{i=1}^n \alpha_i x_i}{1 - \sum_{i=1}^n \alpha_i x_i} \leq \prod_{i=1}^n \left( \frac{1 + x_i}{1 - x_i} \right)^{\alpha_i},$$

where  $\sum_{i=1}^n \alpha_i = 1$  and all  $\alpha_i > 0$ . Equality holds if and only if  $x_1 = x_2 = \dots = x_n$ .

Let  $r_{ik}^\downarrow$  be the  $k$ th largest eigenvalue of  $Z_i$ ,  $i = 1, 2, \dots, m$ , and  $s_k$  be the  $k$ th largest eigenvalue of  $W := \sum_{i=1}^m w_i Z_i$ ,  $k = 1, 2, \dots, n$ . Fan's majorization relation implies

$$\lambda(W) \prec \sum_{i=1}^m w_i \lambda(Z_i), \quad \text{i.e.,} \quad \sum_{k=1}^\ell s_k \leq \sum_{k=1}^\ell \sum_{i=1}^m w_i r_{ik}^\downarrow, \quad \ell = 1, 2, \dots, n.$$

(Note that the components of  $\lambda(\cdot)$  are in nonincreasing order.) Now the convexity and the monotonicity of the function  $f(t) = \ln \frac{1+t}{1-t}$ ,  $0 \leq t < 1$ , imply (see, e.g., [13, p. 343])

$$\sum_{k=1}^n \ln \frac{1+s_k}{1-s_k} \leq \sum_{k=1}^n \ln \frac{1+\sum_{i=1}^m w_i r_{ik}^\downarrow}{1-\sum_{i=1}^m w_i r_{ik}^\downarrow},$$

where equality holds if and only if  $s_k = \sum_{i=1}^m w_i r_{ik}^\downarrow$  for all  $k$ ; that is,  $\lambda(W) = \sum_{i=1}^m w_i \lambda(Z_i)$ . It follows that all  $Z_i$  are simultaneously unitarily diagonalizable with their eigenvalues on the main diagonals in the same order (nonincreasing, say).

Applying the exponential function to both sides and using Lewent's inequality yield

$$\begin{aligned} \prod_{k=1}^n \frac{1+s_k}{1-s_k} &\leq \prod_{k=1}^n \frac{1+\sum_{i=1}^m w_i r_{ik}^\downarrow}{1-\sum_{i=1}^m w_i r_{ik}^\downarrow} \\ (10) \quad &\leq \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1+r_{ik}^\downarrow}{1-r_{ik}^\downarrow} \right)^{w_i} \\ &= \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1+r_{ik}}{1-r_{ik}} \right)^{w_i}, \end{aligned}$$

in which equality occurs in the second inequality if and only if  $r_{1k} = r_{2k} = \dots = r_{mk}$  for  $k = 1, 2, \dots, n$ . Thus both equalities in (10) hold if and only if  $Z_1 = Z_2 = \dots = Z_m$ .

By (4), we have

$$(11) \quad \frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \left( \frac{1+s_k}{1-s_k} \right).$$

Combining (10) and (11) gives the second inequality of (9).

Note that the inequalities in (10) reverse by taking reciprocals, which implies

$$(12) \quad \prod_{k=1}^n \frac{1-s_k}{1+s_k} \geq \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1-r_{ik}}{1+r_{ik}} \right)^{w_i}.$$

Then by (3), we have

$$(13) \quad \frac{\det(I - (\sum_{i=1}^m w_i Z_i)^2)}{|\det(I - U \sum_{i=1}^m w_i Z_i)|^2} \geq \prod_{k=1}^n \left( \frac{1-s_k}{1+s_k} \right).$$

Combining (12) and (13) yields the first inequality of (9).

If either equality holds in (9), then all  $Z_i$  are equal to  $Z$ , say. The conclusions are immediate from Theorem 3.  $\square$

**Remark 6.** Interestingly, it is observed in [4] that the Lewent's inequality follows directly from the convexity of  $f(t) = \ln \frac{1+t}{1-t}$ ,  $0 \leq t < 1$ , applied to the Jensen's inequality. Note that equality occurs in the Lewent's inequality if and only if all the variables  $x_1, x_2, \dots, x_n$  are identical (see, e.g., [1, Theorem 2, p. 3]).

The absolute value of a complex matrix  $Z$ , denoted by  $|Z|$ , is the positive semidefinite square root of  $Z^*Z$ . The following result extends the second inequality in (1).

**Corollary 7.** *Let  $Z_i$ ,  $i = 1, 2, \dots, m$ , be  $n \times n$  complex matrices with singular values  $r_{ik}$  such that  $0 \leq r_{ik} < 1$ ,  $k = 1, 2, \dots, n$ . Then for any  $n \times n$  unitary matrix  $U$*

$$\frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - U \sum_{i=1}^m w_i |Z_i|)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i},$$

where  $w_i > 0$ ,  $i = 1, 2, \dots, m$ , such that  $\sum_{i=1}^m w_i = 1$ . Equality occurs if and only if all  $Z_i$  have the same absolute value, say  $Z$ , and  $\text{Spec}(UZ) = \text{Spec}(Z)$  (in which  $U = I$  if  $Z$  is nonsingular).

*Proof.* With  $(\sum_{i=1}^m w_i |Z_i|)^2 \leq \sum_{i=1}^m w_i |Z_i|^2$  (see, e.g., [12, p. 5]), use Theorem 5.  $\square$

In view of (9), it is tempting to have the lower bound inequality

$$\prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - U \sum_{i=1}^m w_i |Z_i|)|^2}.$$

However, this is not true. Set  $m = n = 2$ ,  $w_1 = w_2 = 1/2$  and take

$$Z_1 = \begin{pmatrix} 0.34 & -0.15 \\ -0.15 & 0.07 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0.02 & -0.01 \\ -0.01 & 0.01 \end{pmatrix}, \quad U = \begin{pmatrix} -0.60 & 0.80 \\ 0.80 & 0.60 \end{pmatrix}.$$

One may check that the left hand side is 0.6281, while the right hand side is 0.6250.

The unitary matrix in Theorem 1 seems superfluous, as one could replace  $Z$  with  $U^*Z$  and leave the singular values unchanged. So perhaps a more natural extension of Theorem 1 to more matrices is giving an upper bound and lower bound, in terms of the singular values of individual matrices, for the quantity  $\frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - \sum_{i=1}^m w_i Z_i)|^2}$ , where  $Z_i$ ,  $i = 1, 2, \dots, m$ , are general contractive matrices. We would guess

$$(14) \quad \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 - r_{ik}}{1 + r_{ik}} \right)^{w_i} \leq \frac{\det(I - \sum_{i=1}^m w_i Z_i^* Z_i)}{|\det(I - \sum_{i=1}^m w_i Z_i)|^2} \leq \prod_{k=1}^n \prod_{i=1}^m \left( \frac{1 + r_{ik}}{1 - r_{ik}} \right)^{w_i}.$$

The first inequality in (14) is untrue in general as it is disproved by substituting  $Z_1$  and  $Z_2$  in (14) with  $U|Z_1|$  and  $U|Z_2|$ , respectively, in the previous example. However, simulation seems to support the second inequality which is unconfirmed yet.

**Acknowledgments.** The work was partially supported by National Natural Science Foundation of China (NNSFC) No. 11601314 and No. 11571220.

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